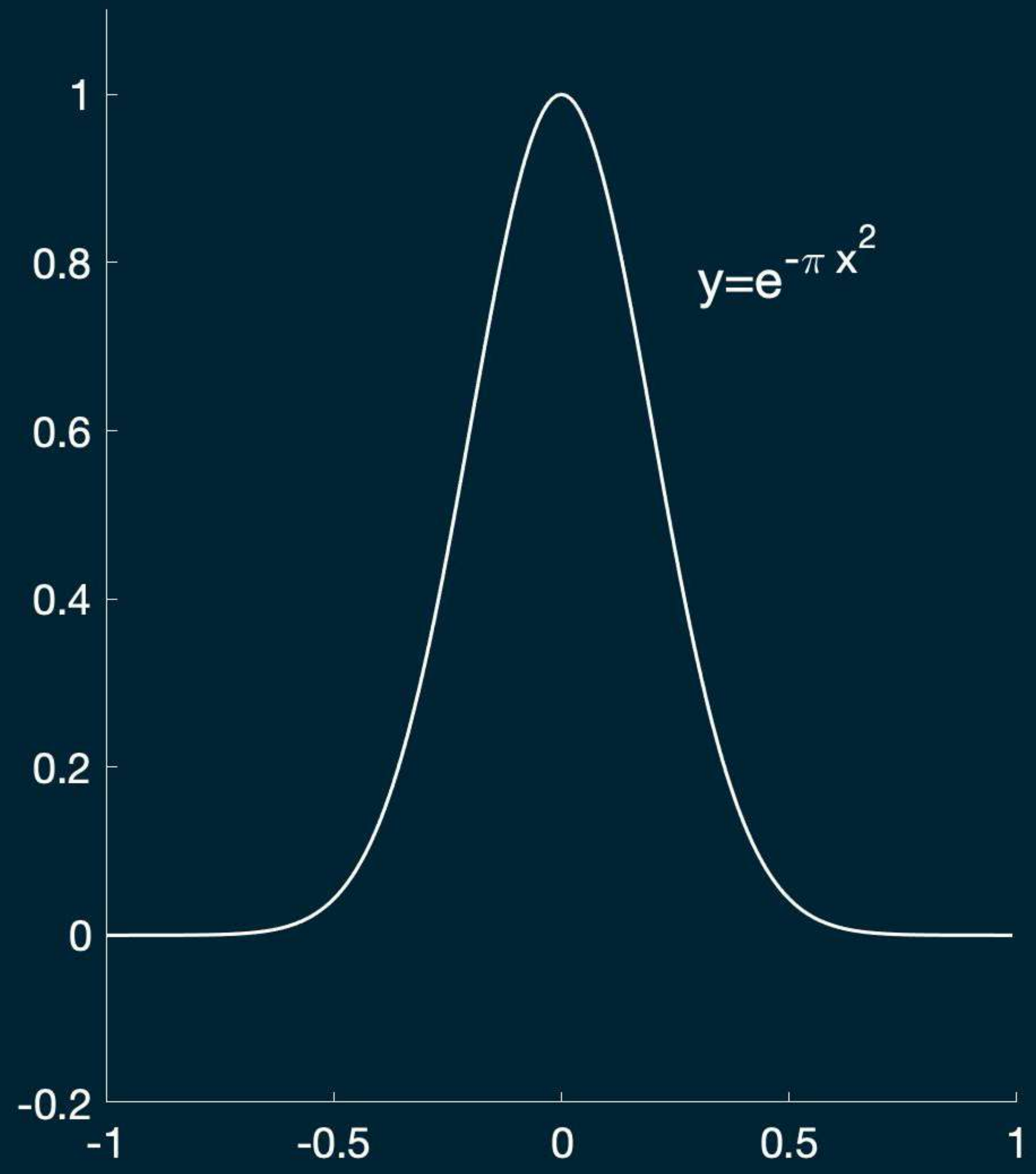
Algorithms and Foundational Math

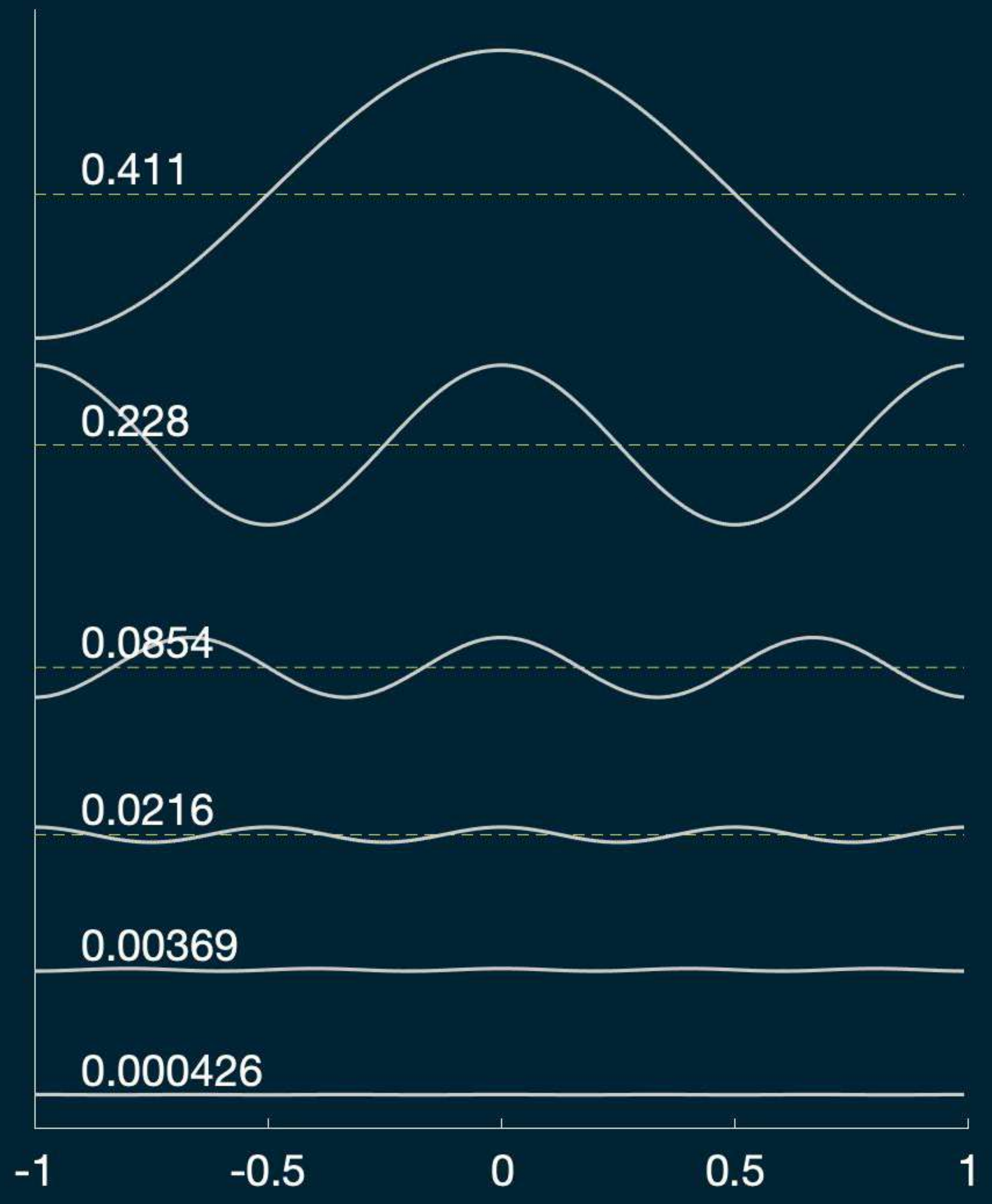
Part 1b

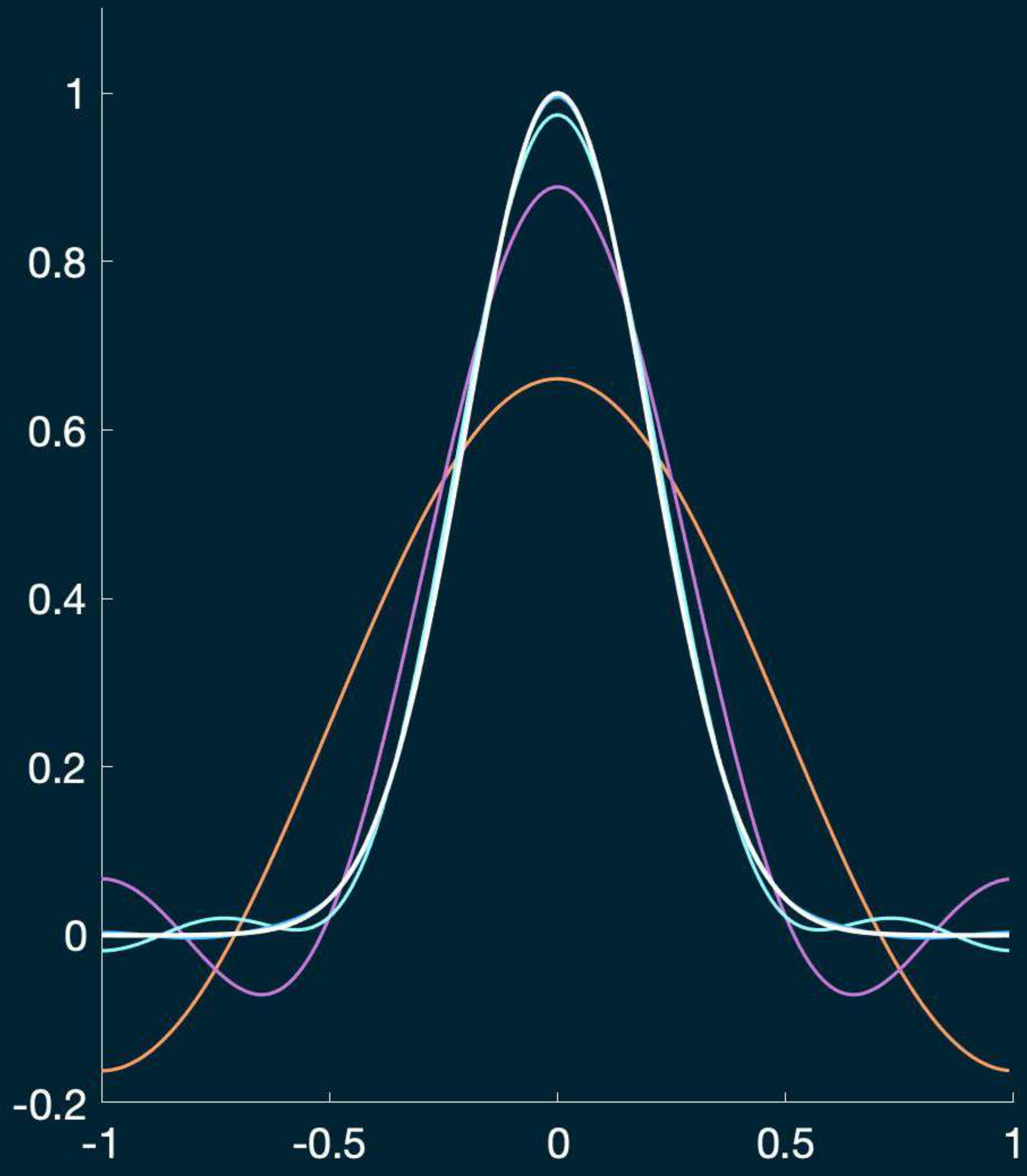
The Fourier transform in one dimension

Fourier reconstruction of a Gaussian function

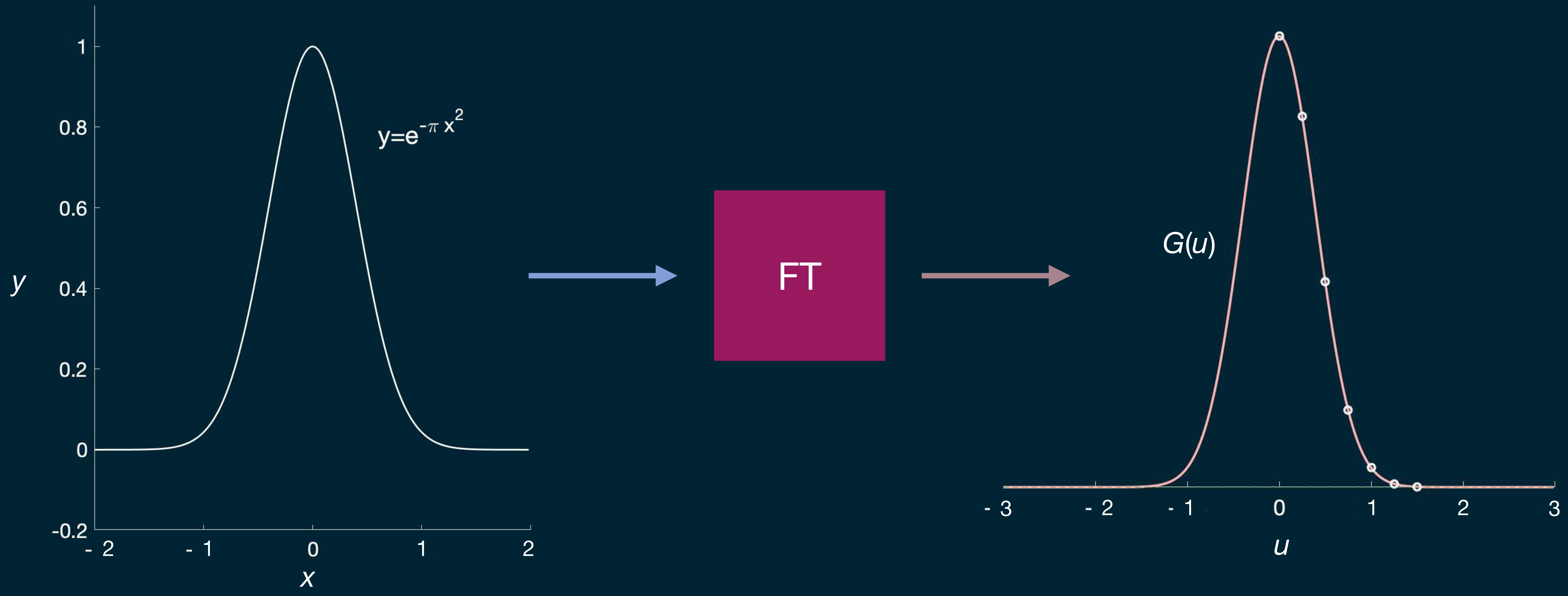


“Converged” at 6 terms





The Fourier Transform gives us the coefficients



The formulas

Fourier transform

$$G(u) = \int g(x) e^{-i2\pi ux} dx$$

Inverse Fourier transform

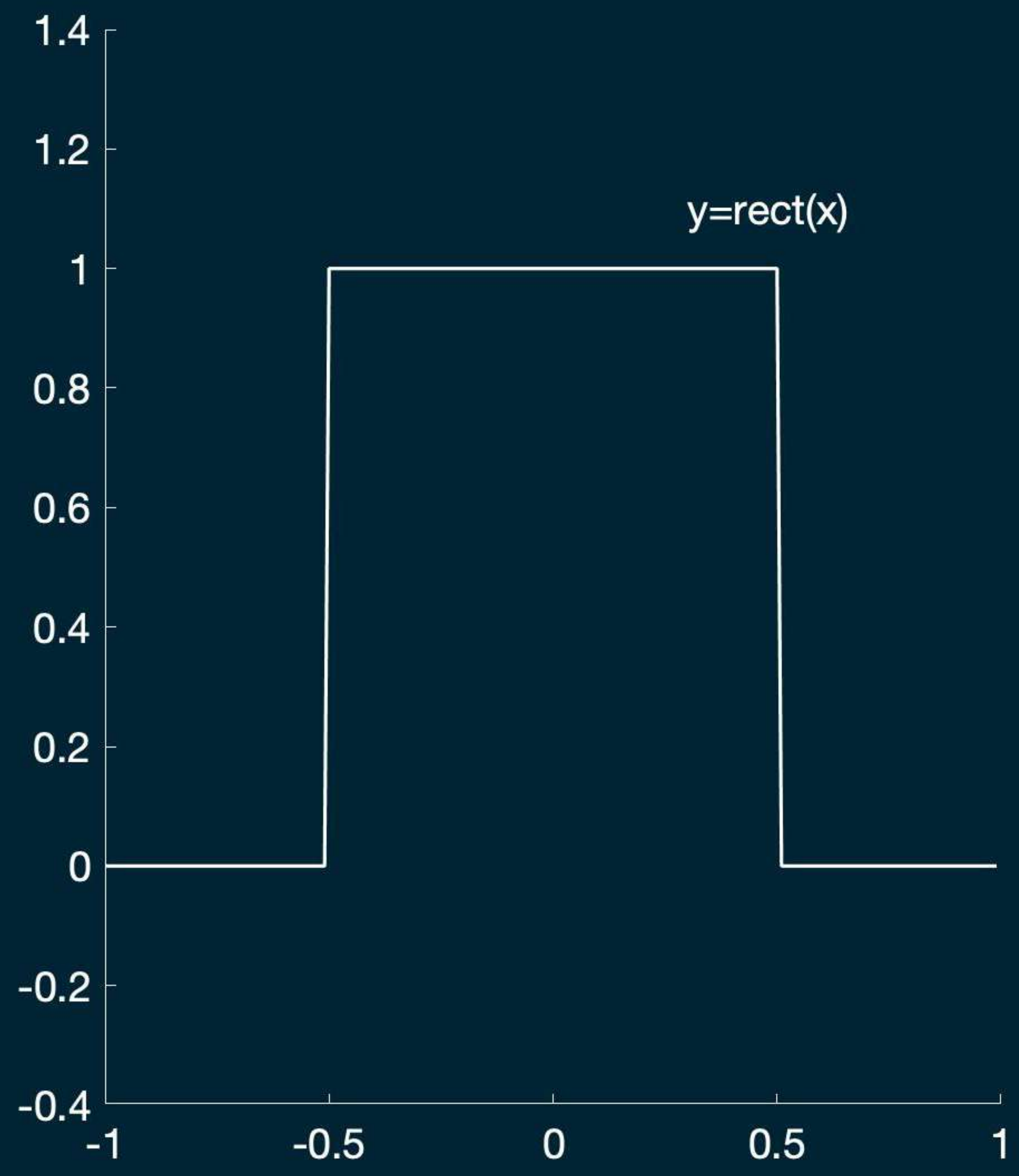
$$g(x) = \int G(u) e^{+i2\pi ux} du$$

Example:

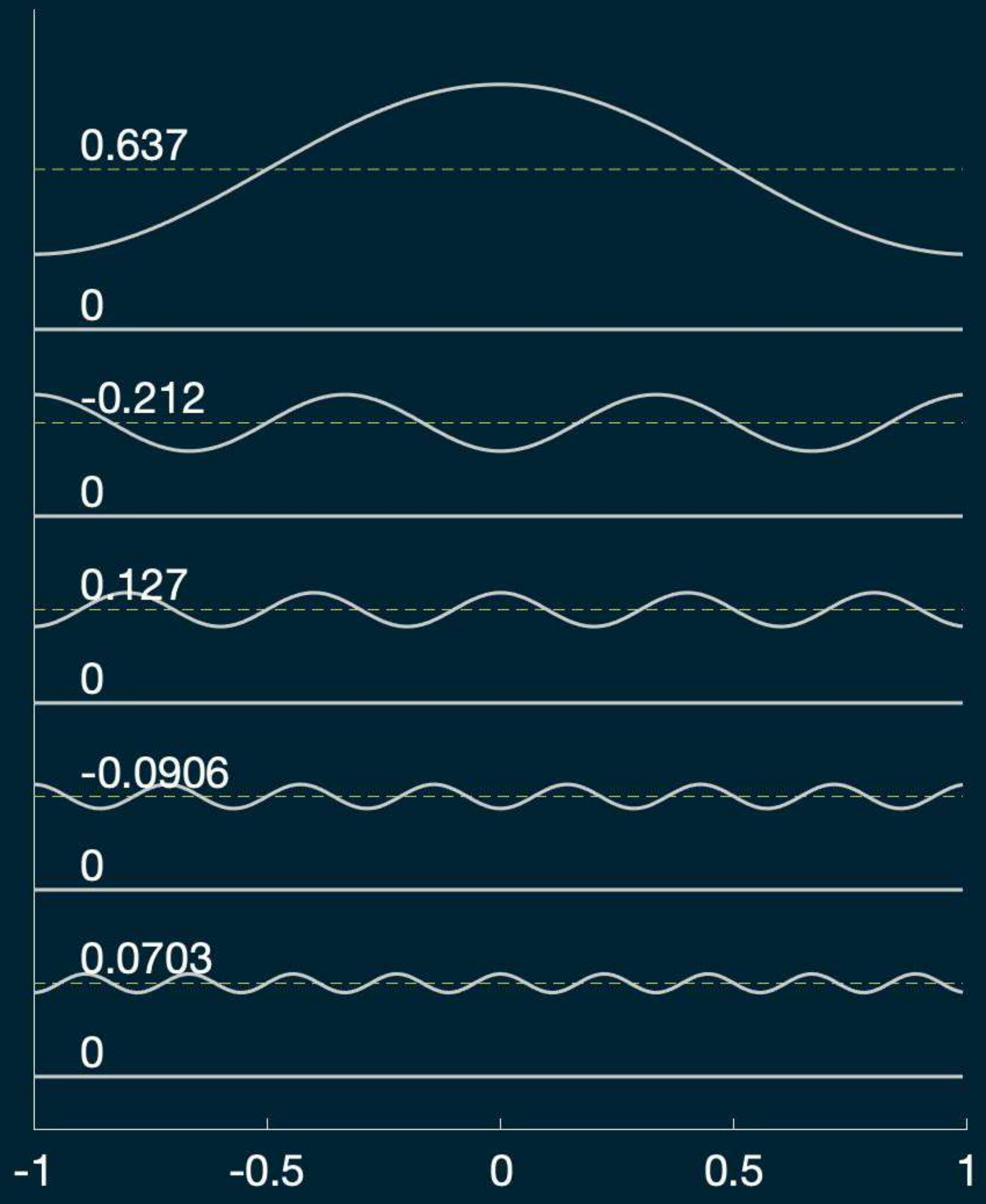
$$g(x) = e^{-\pi x^2}$$

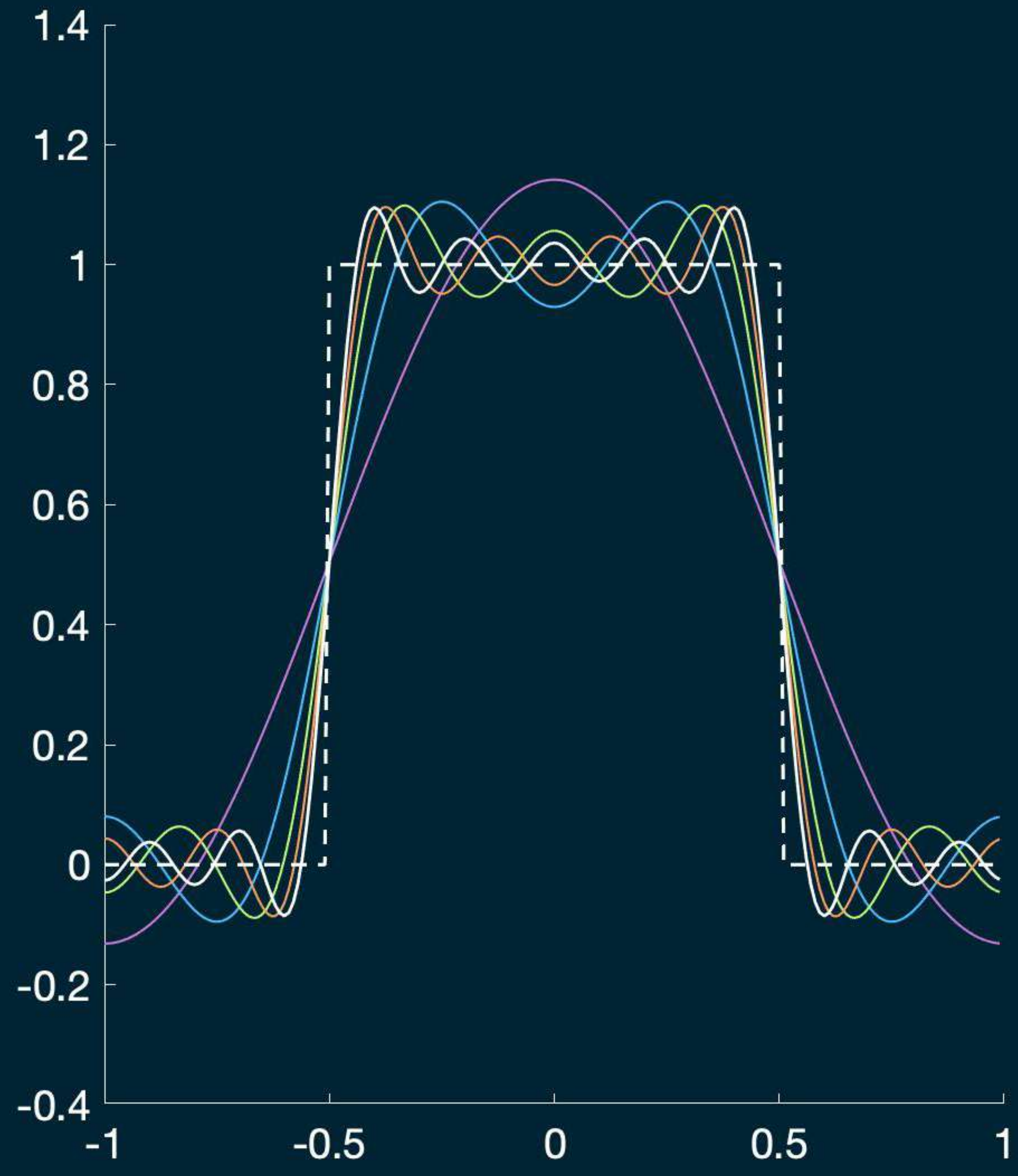
$$G(u) = e^{-\pi u^2}$$

Fourier reconstruction of a rectangular function

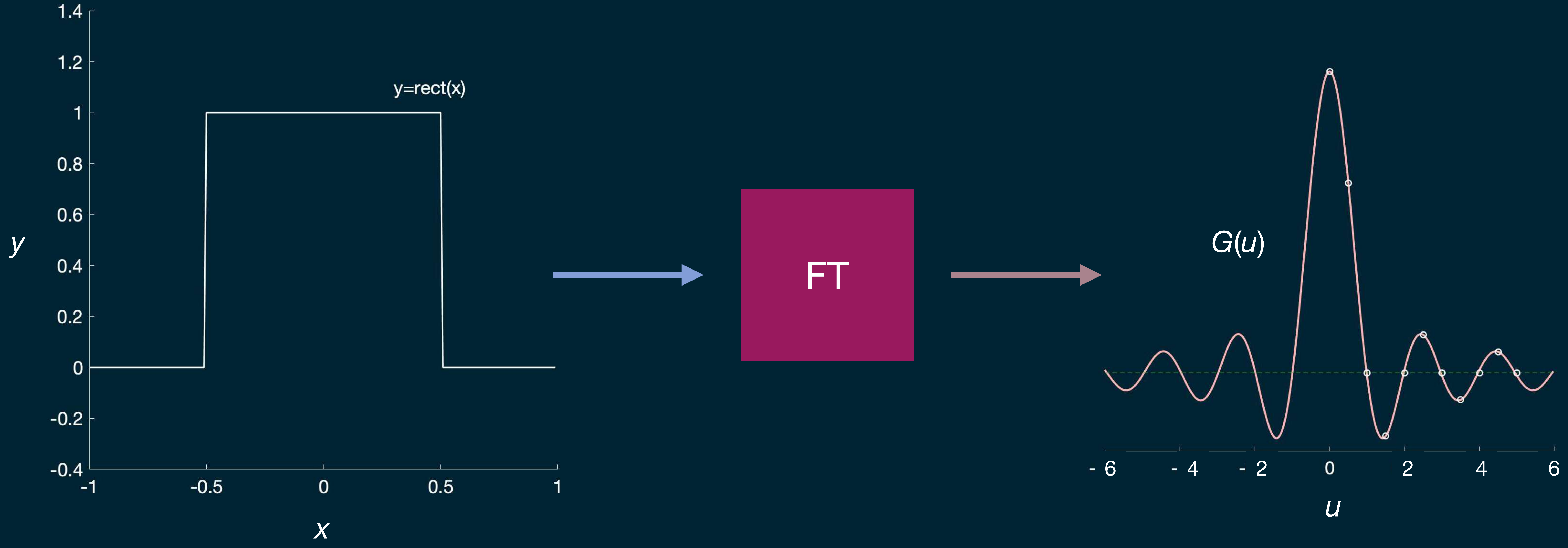


Nowhere near convergence at 10 terms





The Fourier Transform of rect(x) is sinc(u)



$$\mathrm{rect}(x) \rightarrow \frac{\sin(\pi u)}{\pi u}$$

$\frac{\sin(\pi u)}{\pi u}$ is also known as: $\mathrm{sinc}(u)$

Fourier transform pairs

$$e^{-\pi x^2} \rightarrow e^{-\pi u^2}$$

$$\mathrm{rect}(\mathrm{x}) \rightarrow \frac{\sin(\pi \mathrm{u})}{\pi \mathrm{u}}$$

$$\delta(x) \rightarrow 1$$

1D Fourier transform properties

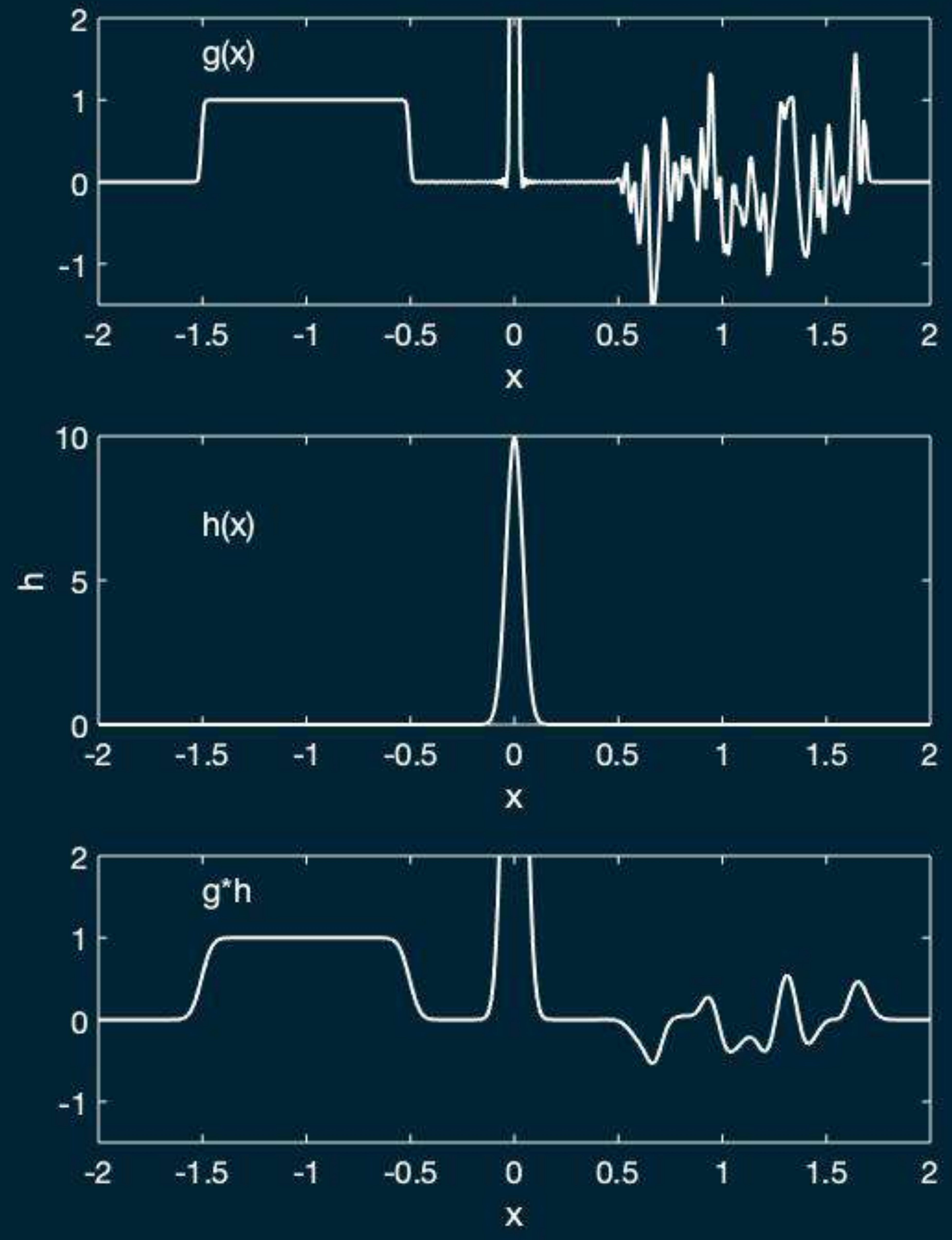
$$g(x) + h(x) \rightarrow G(x) + H(x)$$ **Linearity**

$$ag(ax) \rightarrow G(u/a)$$ **Scale**

$$g(x-b) \rightarrow G(u)e^{-i2\pi ub}$$ **Shift**

$$g \star h \rightarrow G(u)H(u)$$ **Convolution**

Convolution with a Gaussian kernel

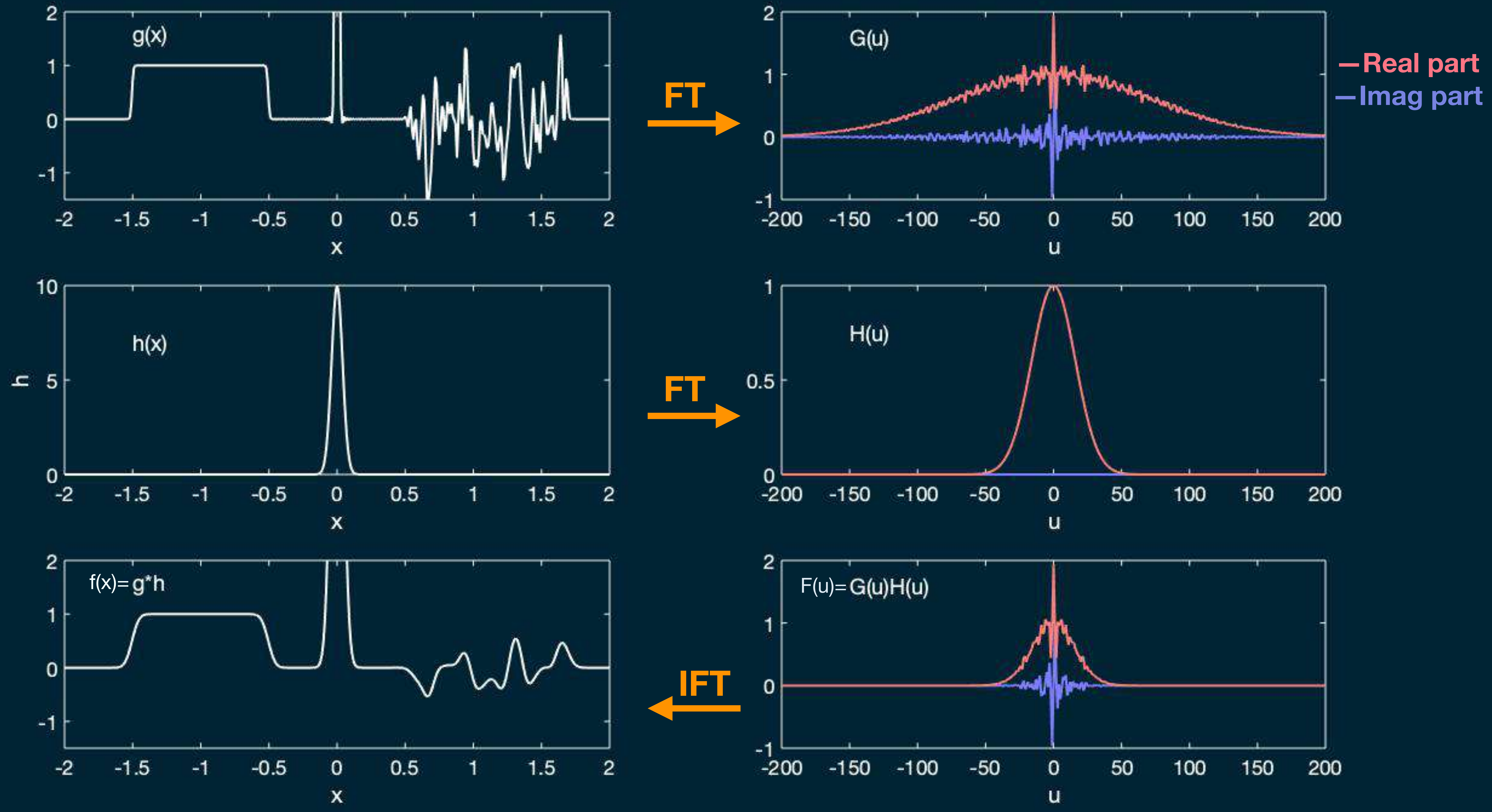


<u>Convolution</u>

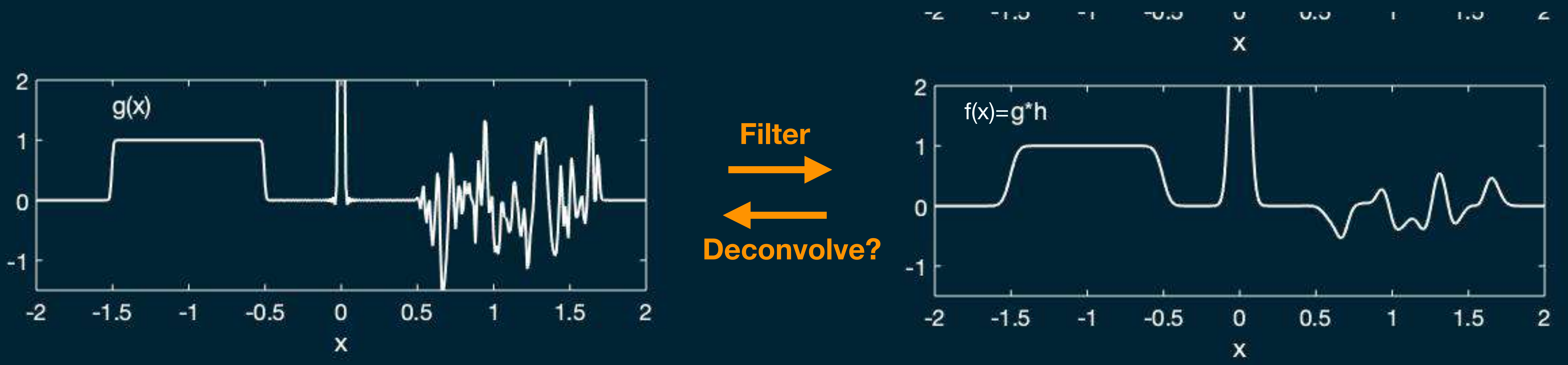
$f(x) = g \star h$ means:

$$f(x) = \int g(x-s)h(s)ds$$

Convolution with a Gaussian kernel



What about de-convolution?

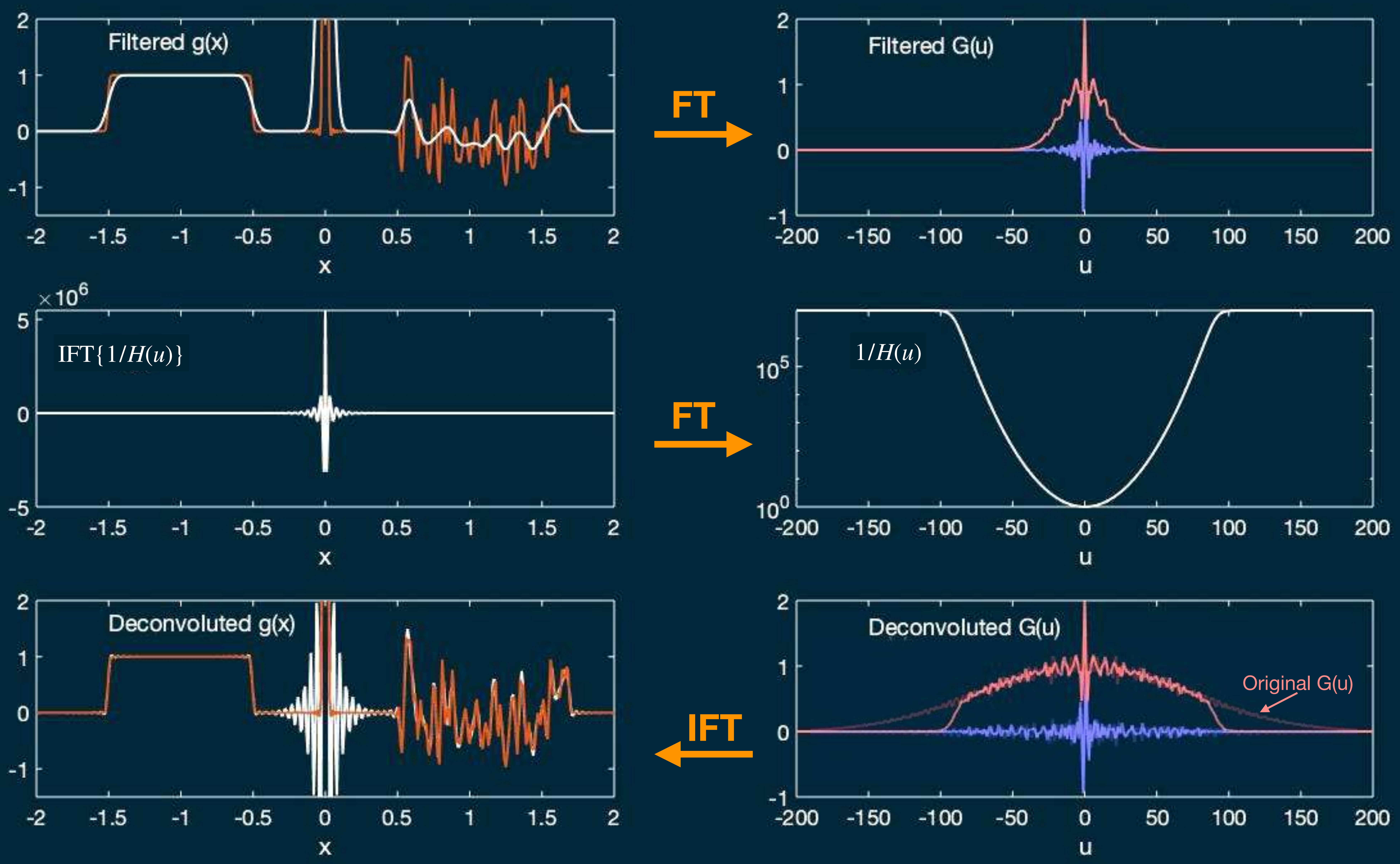


If $F(u) = G(u)H(u)$, shouldn't we be able to recover g, or at least a good approximation $g' \approx g$ by just dividing by H?

That is,

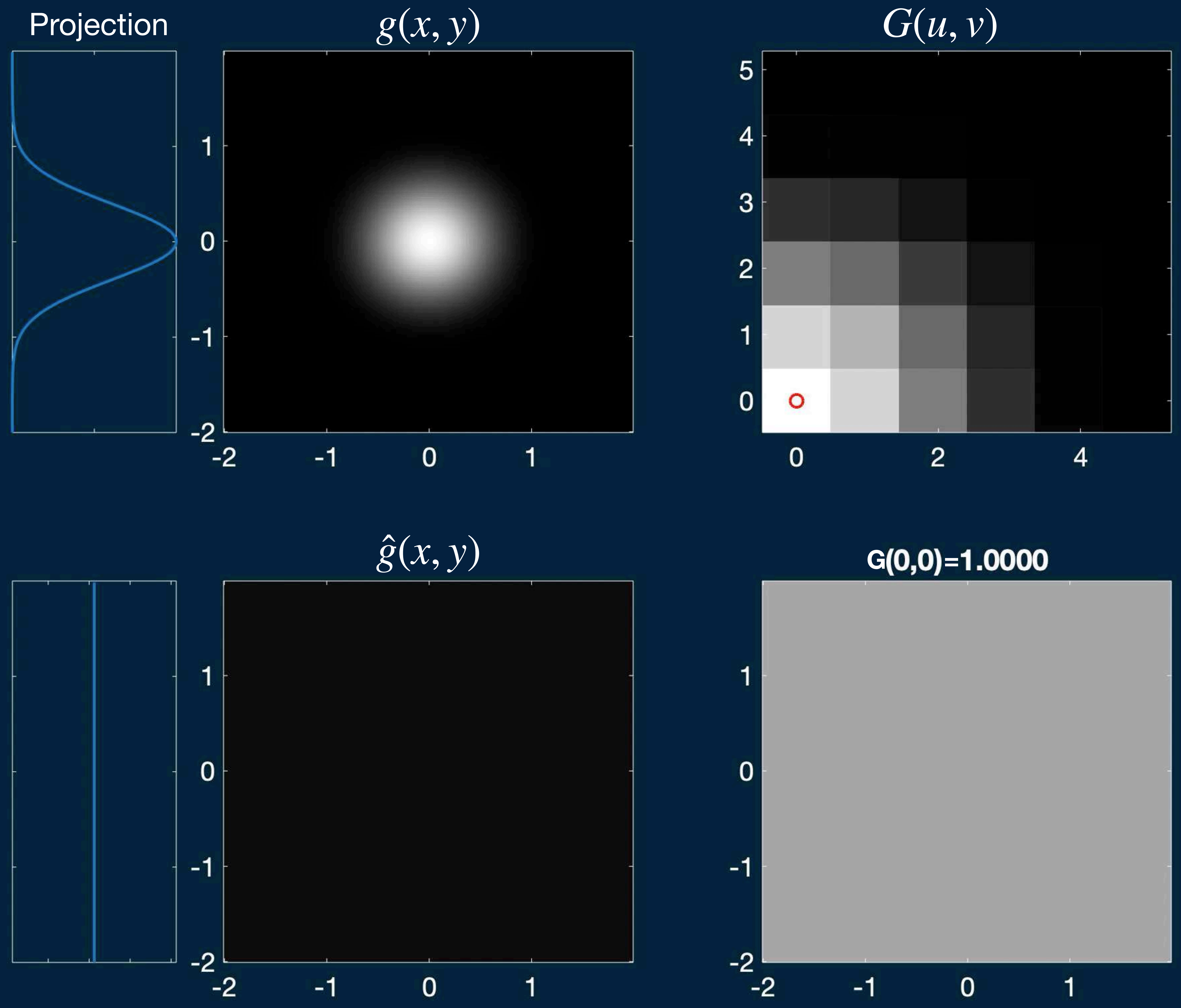
$$G'(u) = \frac{F(u)}{H(u)} \quad \text{and} \quad g'(x) \overset{\text{IFT}}{\leftarrow} G'(u)$$

Deconvolution—the danger is dividing by small numbers

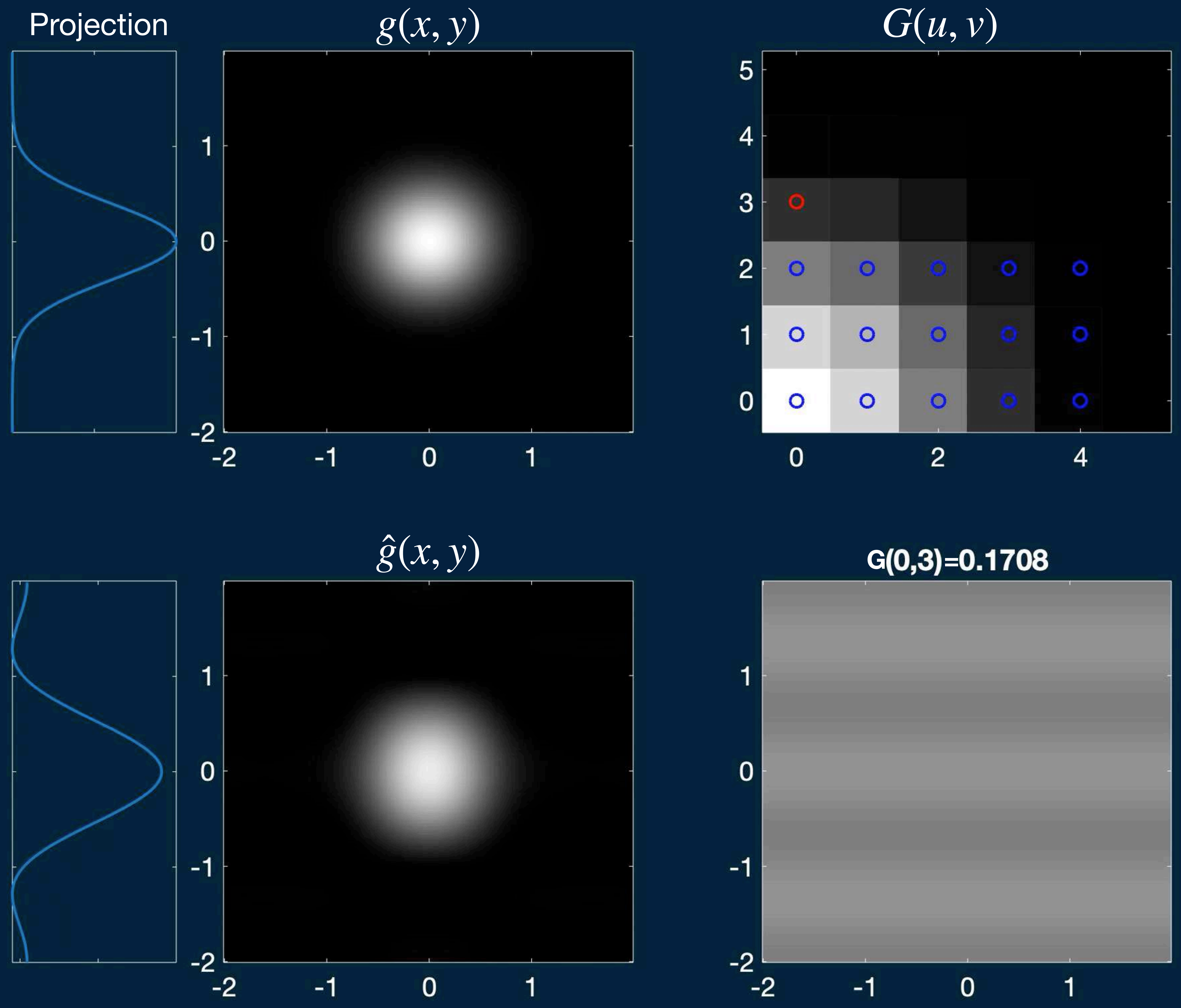


The Fourier transform in two dimensions

Fourier reconstruction of a 2D Gaussian function



Fourier reconstruction of a 2D Gaussian function



2D Fourier transform

$$G(u, v) = \int\int g(x, y)\, e^{-i2\pi(ux+vy)} dx\, dy$$

2D inverse Fourier transform

$$g(x, y) = \int\int G(u, v)\, e^{i2\pi(ux+vy)} du\, dv$$

2D Fourier transform properties

$$ab\, g(ax, by) \rightarrow G(u/a, v/b)$$ Scale

$$g(x-a, y-b) \rightarrow G(u, v)e^{-i2\pi(au+bv)}$$ Shift

$$g * h \rightarrow GH$$ Convolution

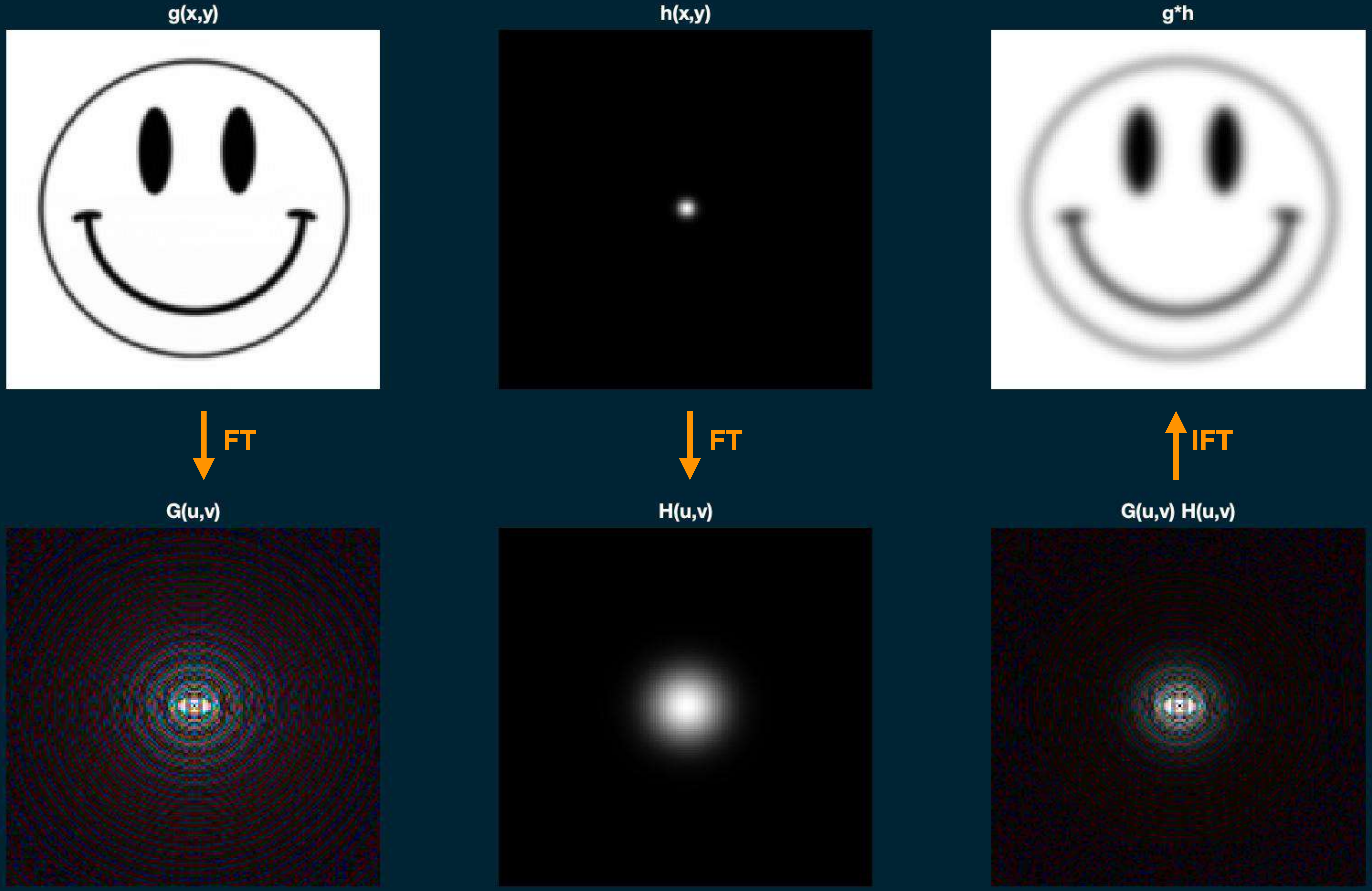
$$g(x', y') \rightarrow G(u', v')$$ Rotation

$$P_y\, g(x, y) \rightarrow G(u, 0)$$ Projection

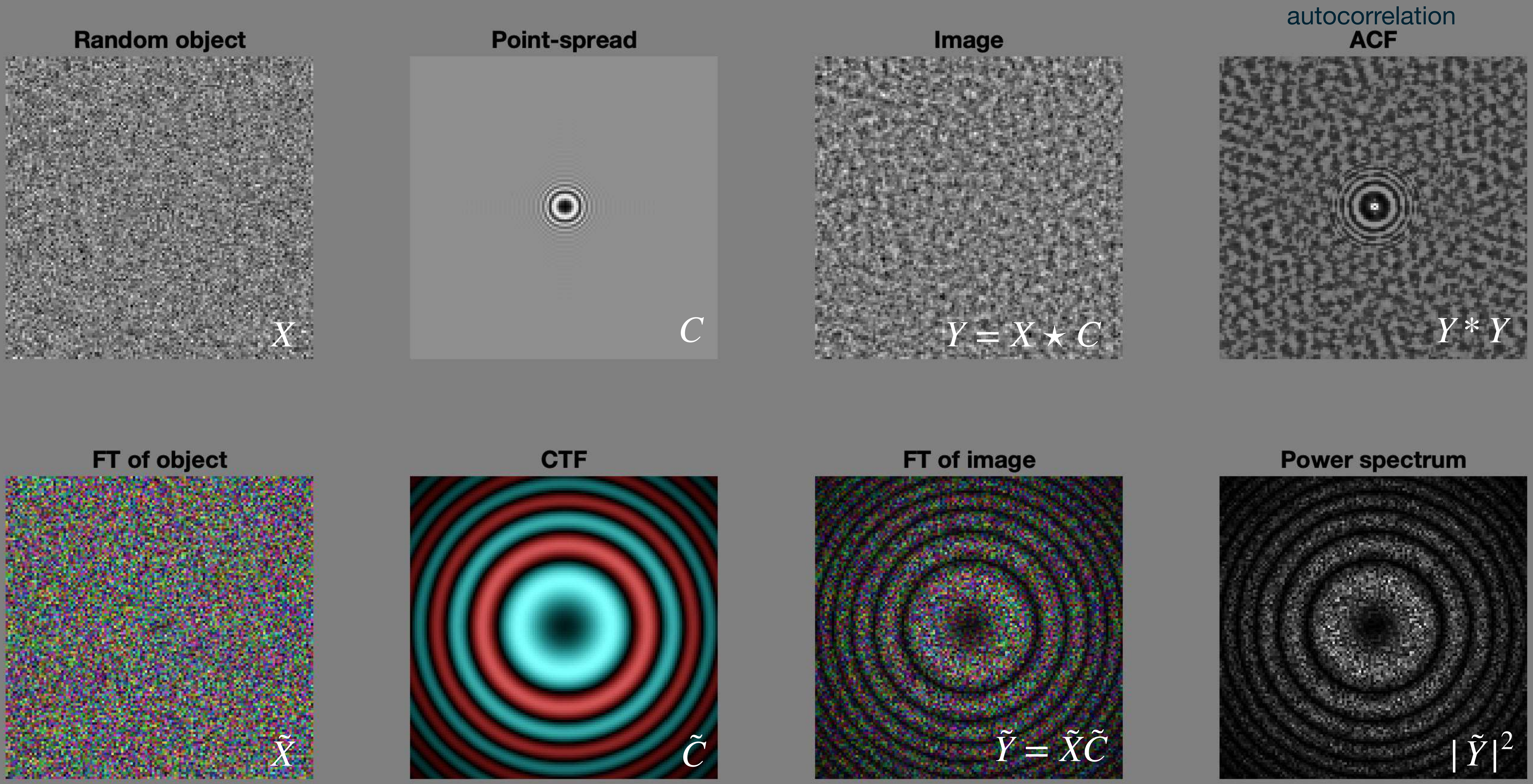
Convolution in 2D

$$G \star H = \iint g(x - s, y - t)\, h(s, t)\, ds\, dt$$

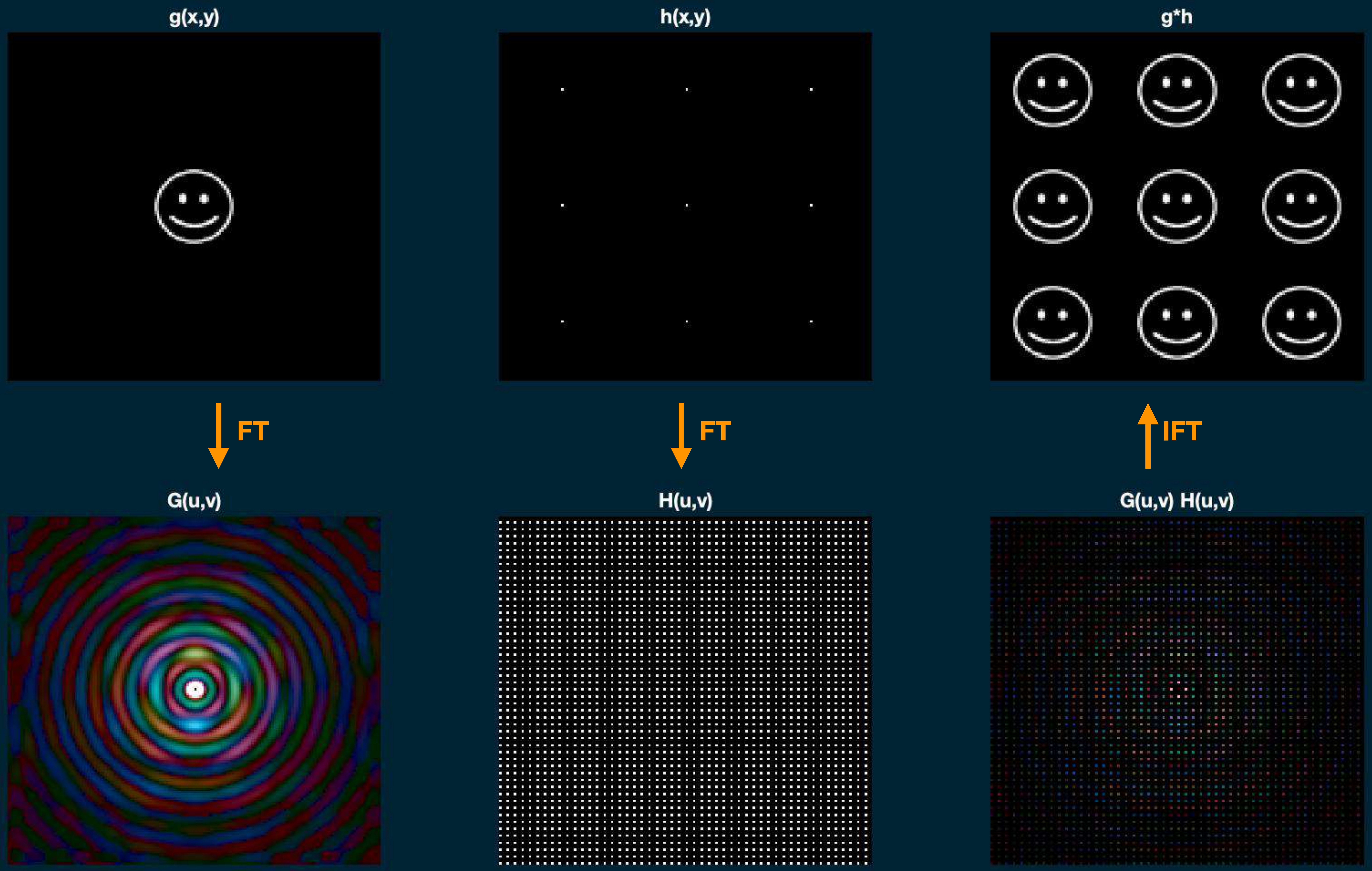
Convolution with a Gaussian



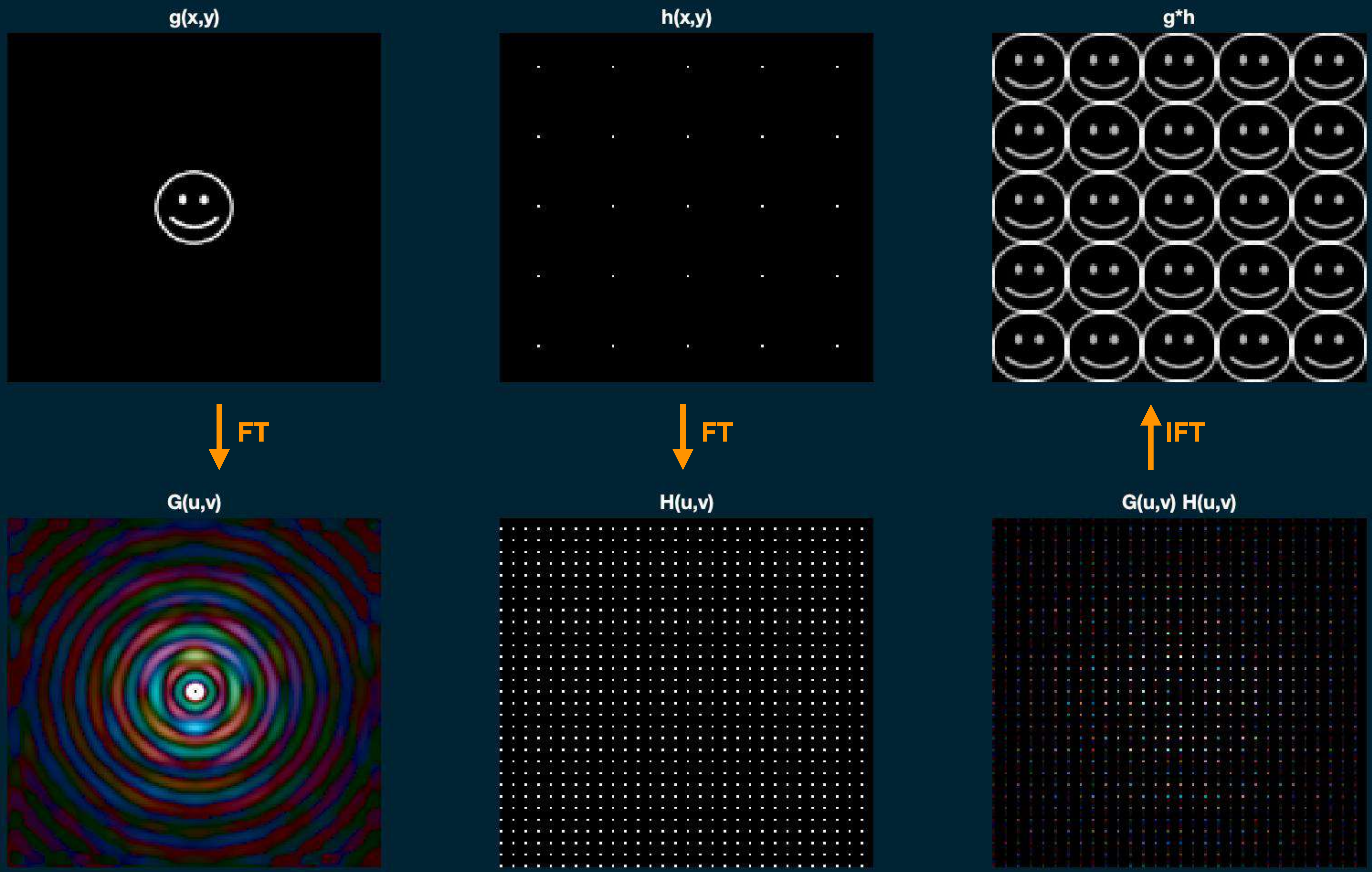
Visualizing the contrast transfer function



Convolution with a lattice



An undersampling lattice



The rotation property

2D Fourier Transform

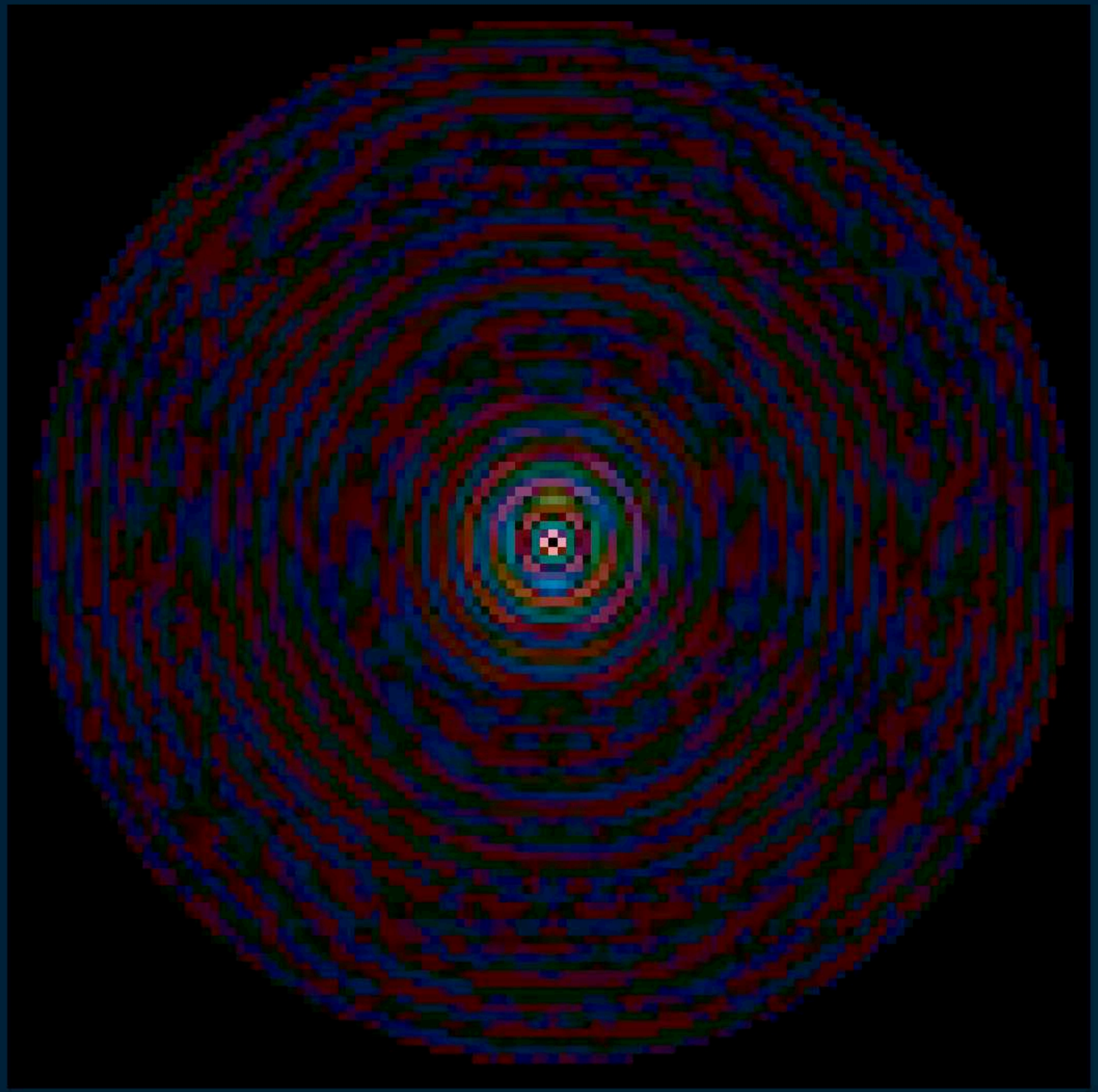
$$G(u,v) = \iint g(x,y) e^{-i2\pi(ux+vy)} dx dy$$

FT using 2D vectors

$$G(\mathbf{u}) = \int\int g(\mathbf{x}) e^{-i2\pi(\mathbf{u}\cdot\mathbf{x})} d^2\mathbf{x}$$

The dot-product is invariant under rotations!

FT $\longrightarrow$



Let R_θ signify a rotation, and

$$(x', y') = R_\theta(x, y)$$

$$(u', v') = R_\theta(u, v)$$

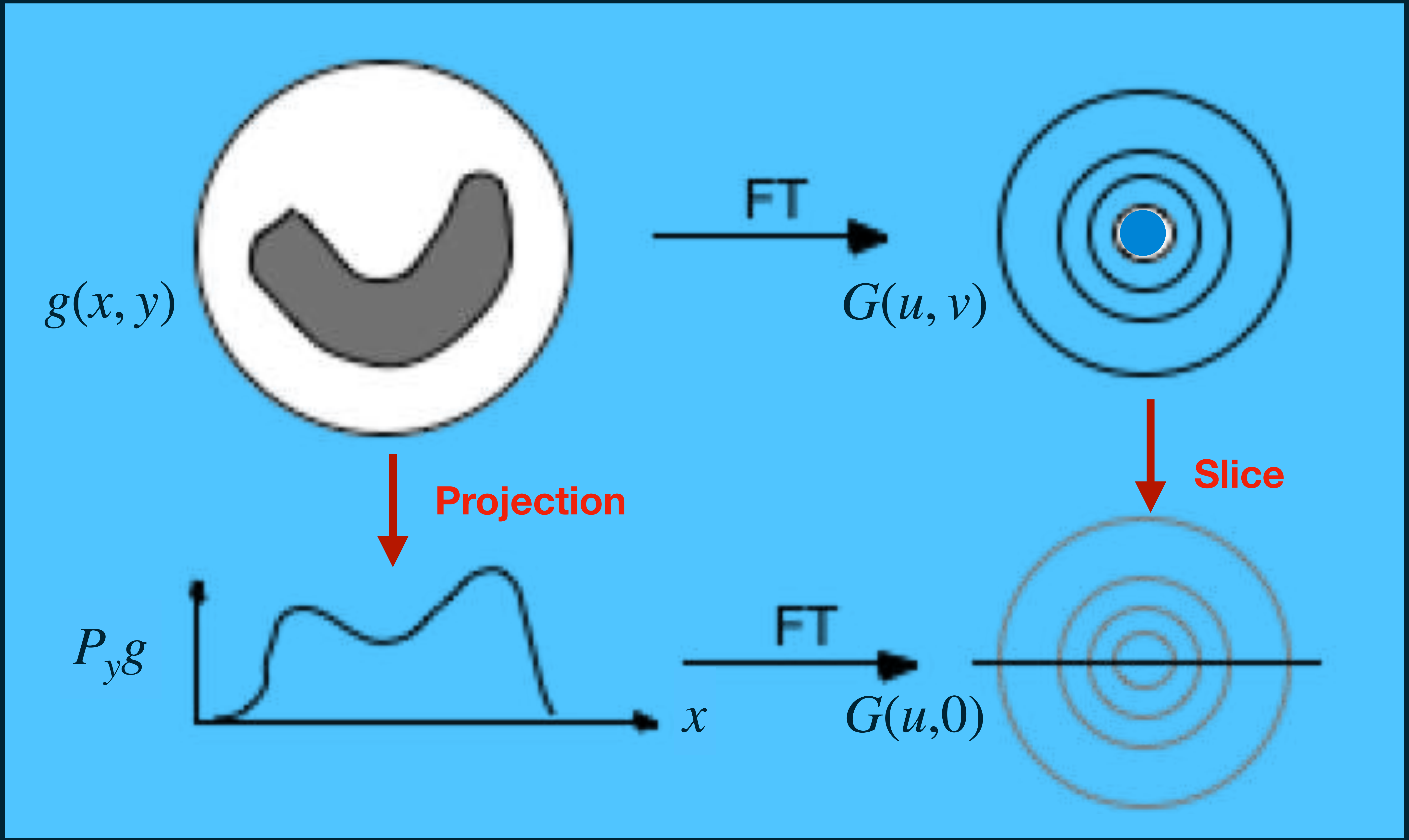
then

$$g(x', y') \to G(u', v')$$

or alternatively,

$$g(R_\theta \mathbf{x}) \to G(R_\theta \mathbf{u})$$

The Fourier Slice Theorem

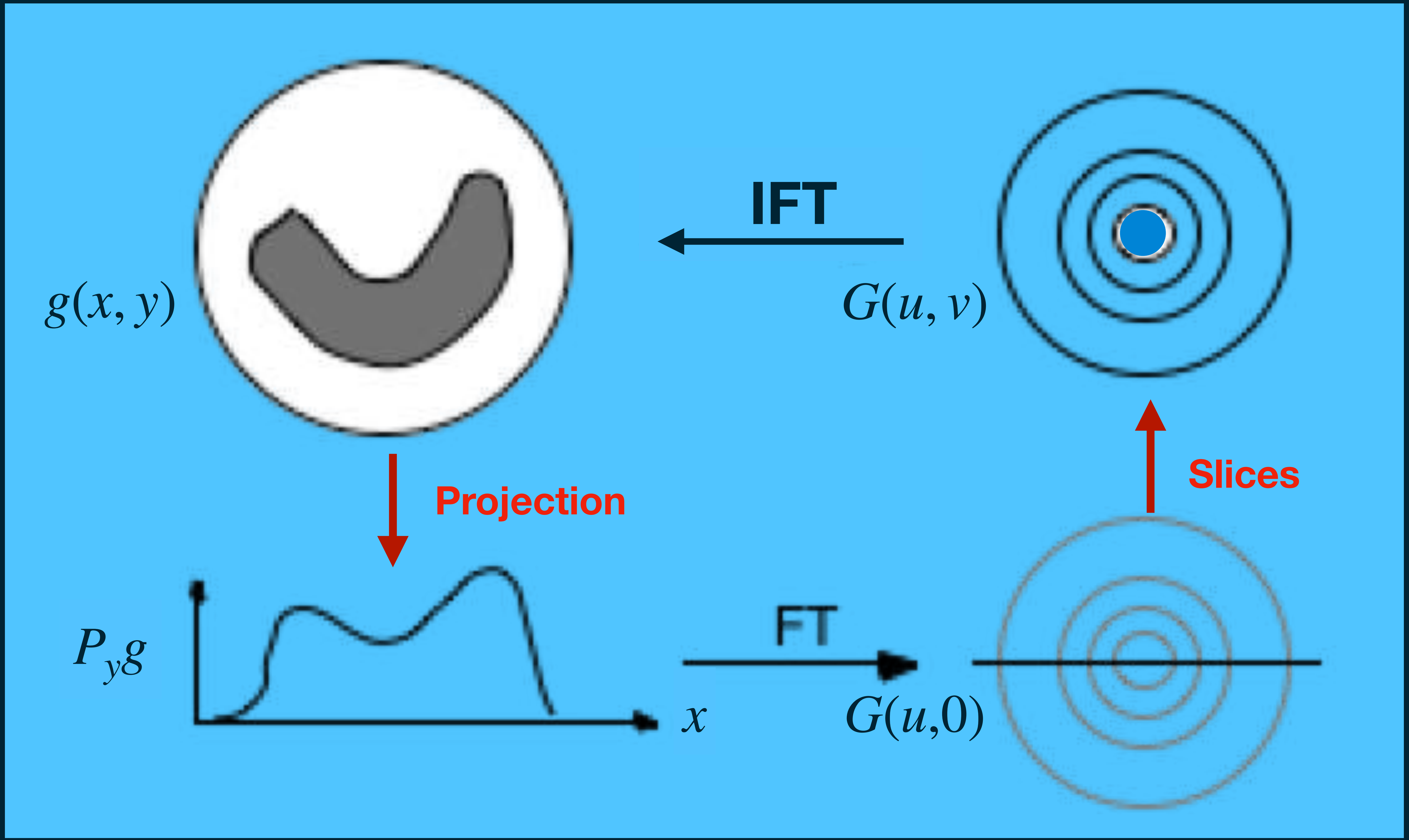


$$G(u,v) = \iint g(x,y) e^{-i2\pi(ux+vy)} dxdy$$

$$G(u,0) = \int \left(\int g(x,y) dy \right) e^{-i2\pi(ux)} dx$$
$$= \mathscr{F}\{P_y g\}$$

$$P_y g(x,y) = \int g(x,y) dy$$

Reconstruction using the Fourier Slice Theorem



$$G(u,v) = \iint g(x,y)e^{-i2\pi(ux+vy)}dxdy$$

$$G(u,0) = \int \left(\int g(x,y)dy \right) e^{-i2\pi(ux)}dx$$
$$= \mathcal{F}\{P_y g\}$$

$$P_y g(x,y) = \int g(x,y)dy$$

The rotation property says:
If we can collect projections from all directions, we can construct all of $G(u,v)$

The discrete FT is what is calculated on a computer

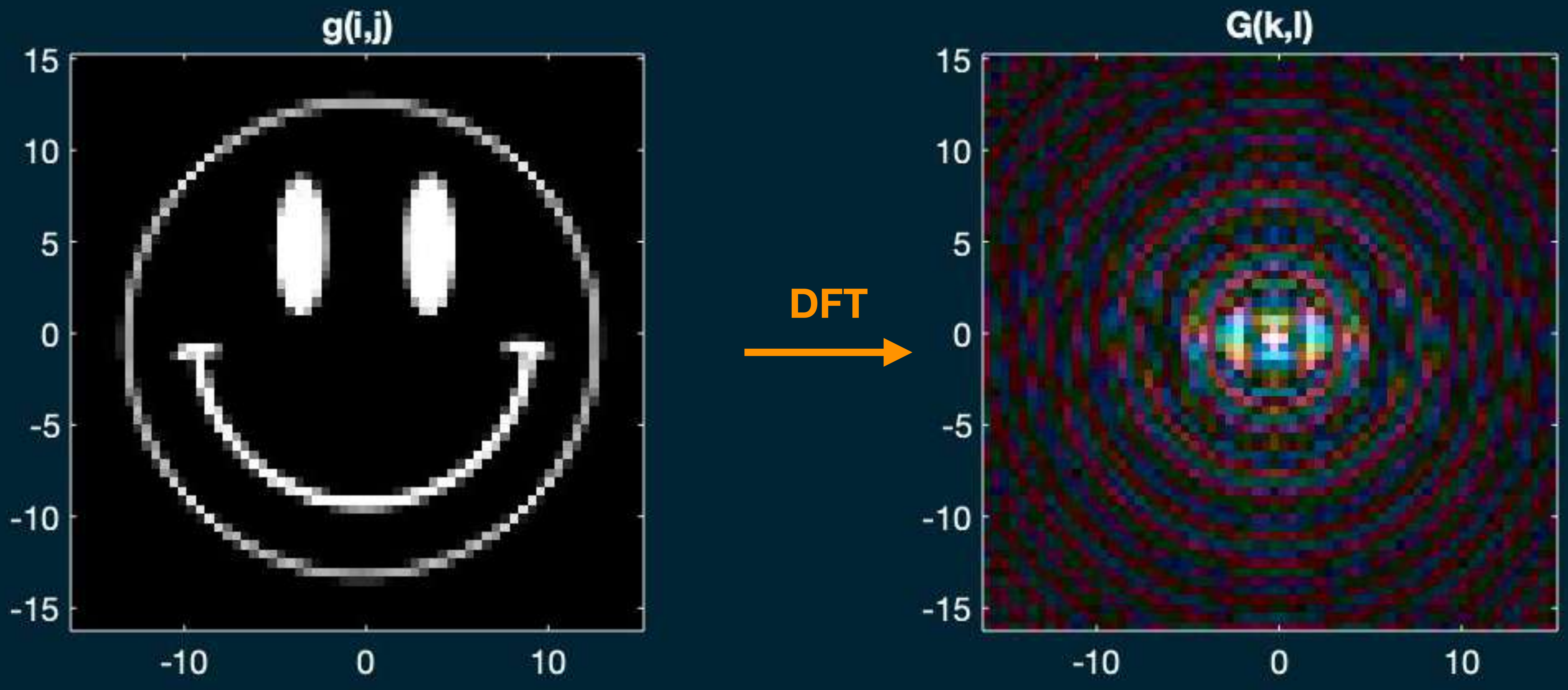
2D Fourier transform

$$G(u,v) = \int\int g(x,y)\, e^{-i2\pi(ux+vy)} dx\, dy$$

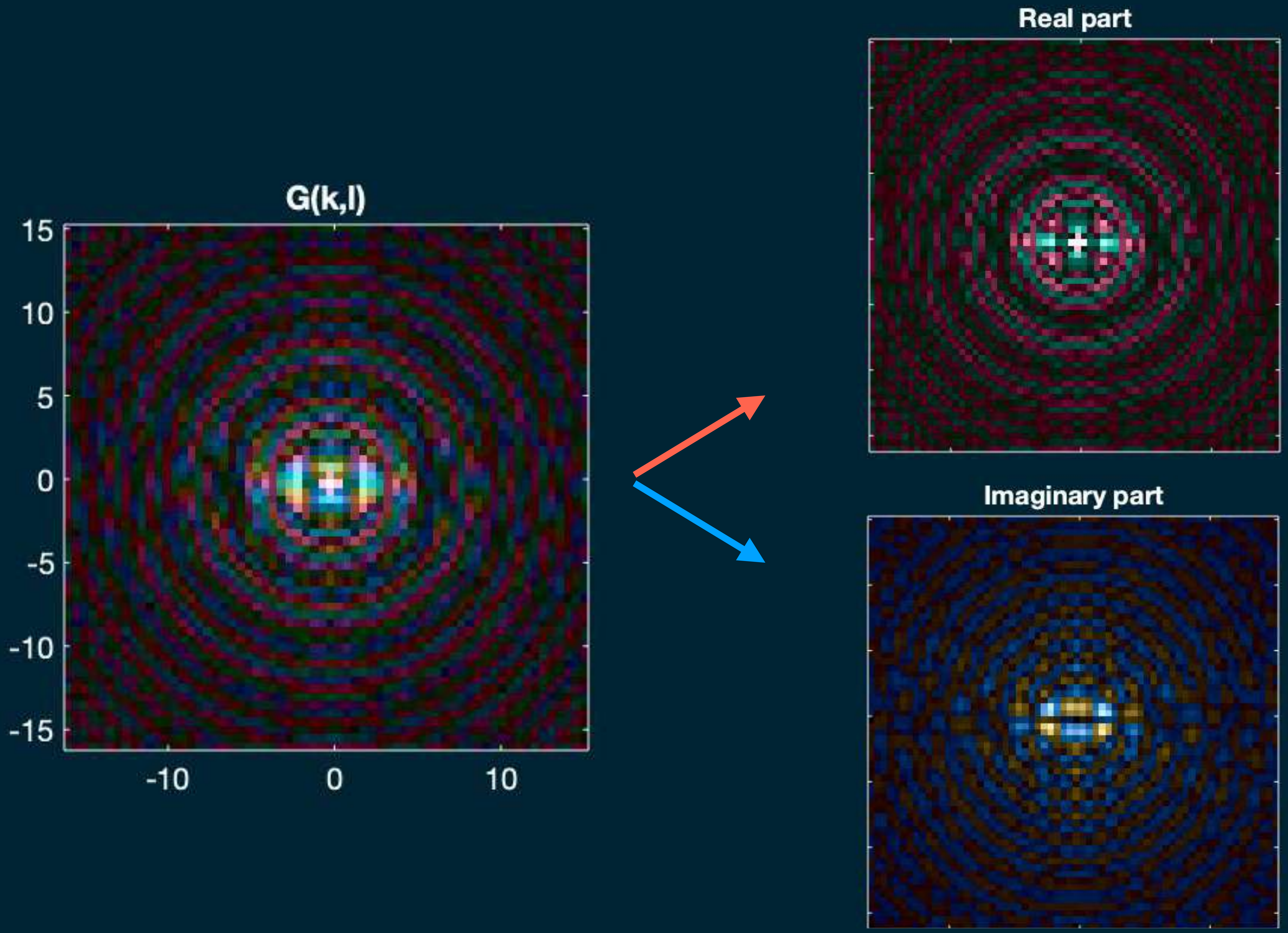
2D discrete Fourier transform

$$G(k,l) = \frac{1}{N} \sum_{i,j=-N/2}^{N/2-1} g(i,j)\, e^{-i2\pi(ik+jl)/N}$$

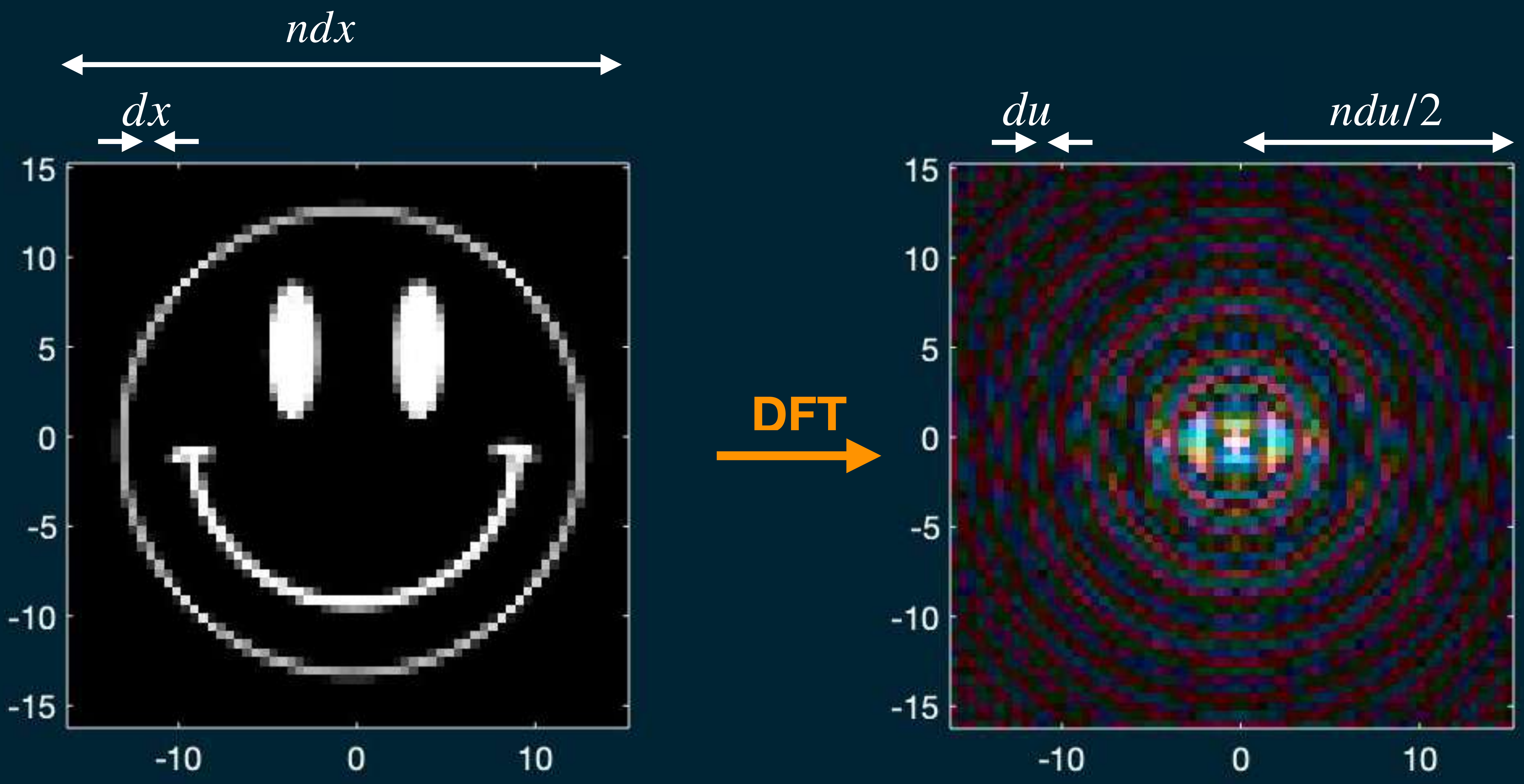
The DFT of a 32 x 32 pixel image has 32 x 32 complex pixel values



But the DFT of a real image has twofold redundancy



What is the pixel size of the transformed image?



$$dx = \frac{1}{n\,du} \qquad ndx = \frac{1}{du} \qquad du = \frac{1}{n\,dx} \qquad ndu = \frac{1}{dx}$$

Note that the sampling frequency $1/dx$ …… corresponds to twice the maximum accessible frequency $ndu/2$.