Cryo-EM Principles

Mathematical Background

Complex numbers
and the complex exponential

Fred Sigworth
Yale University

Why complex numbers?

- Equations are simpler
- Natural for Fourier transforms
- Magnitude and phase of structure factors

i, the imaginary unit

$$i = \sqrt{-1}$$

A complex number

Real part

$$z = a + ib$$

Imaginary part

$$w = c + id$$

Properties of complex numbers

$$z = a + ib$$
$$w = c + id$$

Add $z + w = (a + c) + i(b + d)$

Multiply $zw = (ab - bd) + i(ad + bc)$

Real part $\mathrm{Re}(z) = a$

Imaginary part $\mathrm{Im}(z) = b$

Absolute value $|z| = \sqrt{a^2 + b^2}$

Conjugate $z^* = a - ib$

(Exercise: Show that $zz^* = |z|^2$)

The exponential function e^x

$$e = 2.718...$$

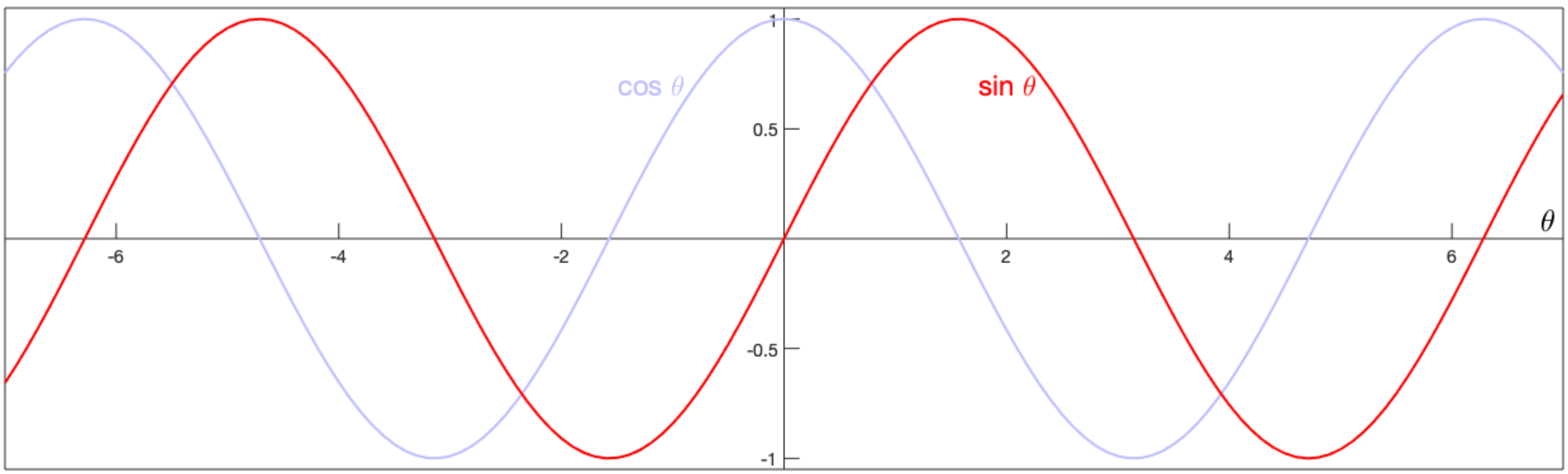
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \times 3} + \ldots$$

An important approximation:

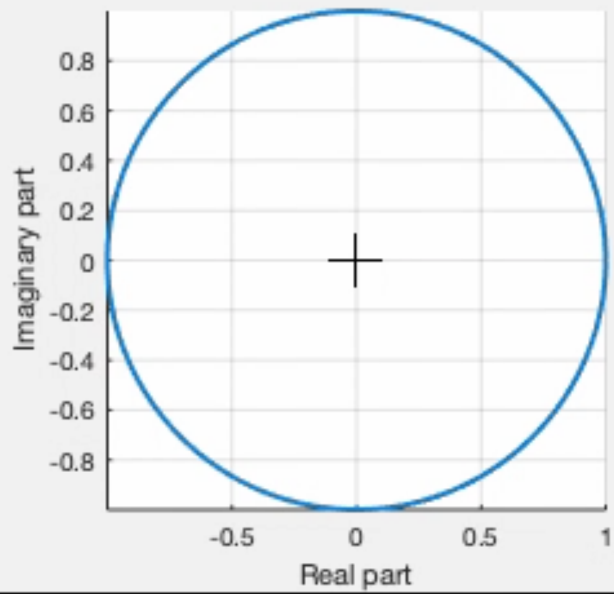
$$e^x \approx 1 + x, \;\; x \ll 1$$

The complex exponential

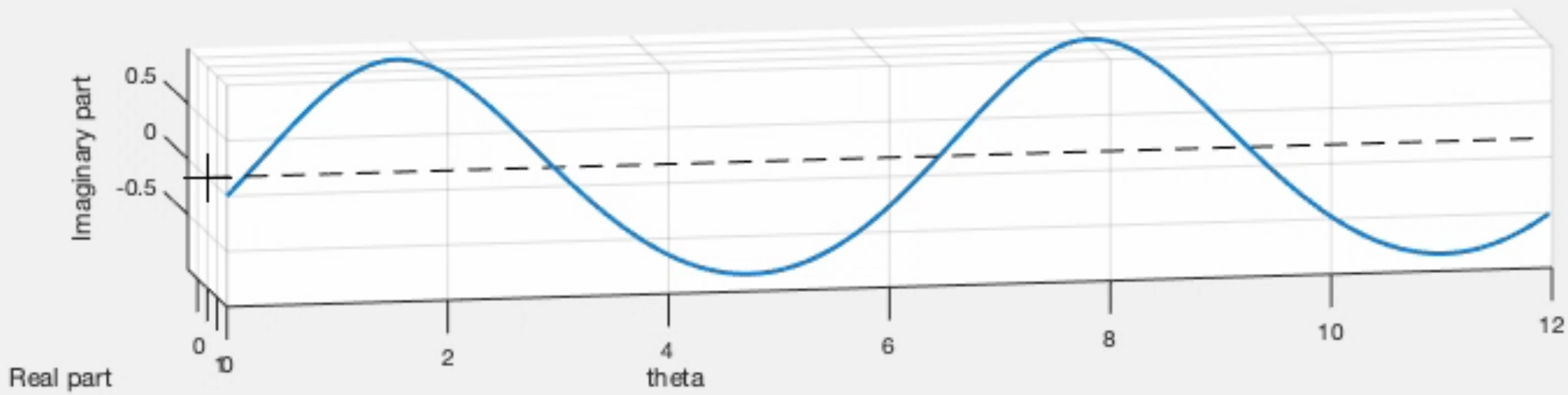
$$e^{i\theta} = \cos\theta + i\sin\theta$$



A plot of $e^{i\theta}$



A plot of $e^{i\theta}$



Any z can be represented as (a, b) or as (r, θ)

$z = a + ib$

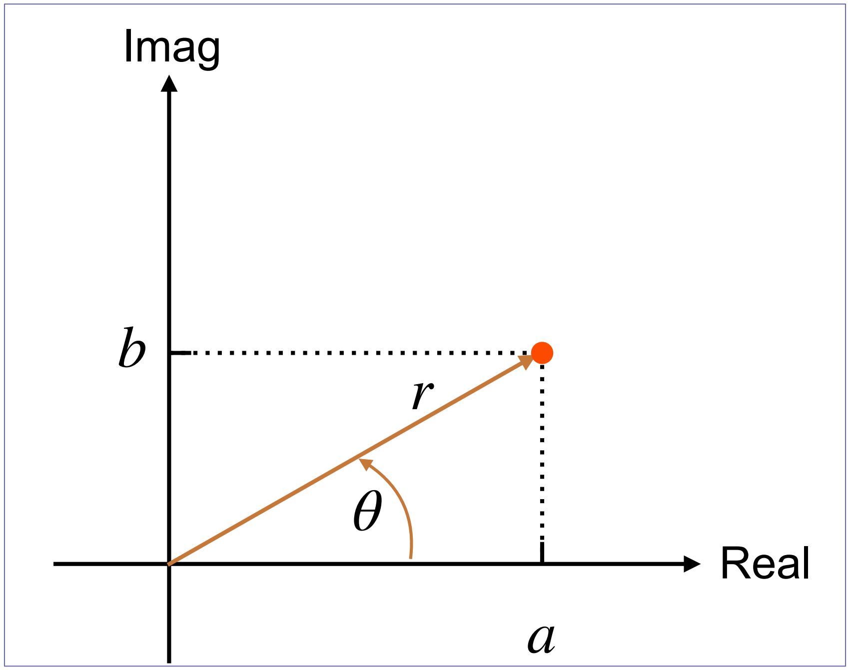
a is the real part

b is the imaginary part

$z = re^{i\theta}$

r is the magnitude

θ is the phase



Recall that

$e^x e^y = e^{x+y}$

so, when you multiply two complex numbers, the phases add:

$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.

Properties of complex numbers

$$z = a + ib$$
$$w = c + id$$

Add $z + w = (a + c) + i(b + d)$

Multiply
$$zw = (a + ib)(c + id)$$
$$= ab + iad + ibc + i^2 bd \quad (i^2 = -1)$$
$$= (ab - bd) + i(ad + bc)$$

Real part $\mathrm{Re}(z) = a$

Imaginary part $\mathrm{Im}(z) = b$

Absolute value $|z| = \sqrt{a^2 + b^2}$

Conjugate $z^* = a - ib$

(Exercise: Show that $zz^* = |z|^2$)