Complex numbers
and the complex exponential

Fred Sigworth
Yale University
Why complex numbers?

- Equations are simpler
- Natural for Fourier transforms
- Magnitude and phase of structure factors
$i$, the imaginary unit

\[ i = \sqrt{-1} \]

A complex number  \( z = a + ib \)

\[ w = c + id \]
### Properties of complex numbers

<table>
<thead>
<tr>
<th>Description</th>
<th>Formula</th>
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</thead>
<tbody>
<tr>
<td><strong>Add</strong></td>
<td>( z + w = (a + c) + i(b + d) )</td>
</tr>
<tr>
<td><strong>Multiply</strong></td>
<td>( zw = (ab - bd) + i(ad + bc) )</td>
</tr>
<tr>
<td><strong>Real part</strong></td>
<td>( \text{Re}(z) = a )</td>
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<tr>
<td><strong>Imaginary part</strong></td>
<td>( \text{Im}(z) = b )</td>
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<td><strong>Absolute value</strong></td>
<td>(</td>
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<tr>
<td><strong>Conjugate</strong></td>
<td>( z^* = a - ib )</td>
</tr>
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(Exercise: Show that \( zz^* = |z|^2 \))
The exponential function $e^x$

\[ e = 2.718... \]

\[ e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \times 3} + \ldots \]

An important approximation:

\[ e^x \approx 1 + x, \quad x \ll 1 \]
The complex exponential

\[ e^{i\theta} = \cos \theta + i \sin \theta \]
A plot of $e^{i\theta}$
A plot of $e^{i\theta}$
Any $z$ can be represented as $(a, b)$ or as $(r, \theta)$

\[ z = a + ib \quad \quad \quad \quad z = re^{i\theta} \]

- $a$ is the real part
- $b$ is the imaginary part
- $r$ is the magnitude
- $\theta$ is the phase

Recall that

\[ e^x e^y = e^{x+y} \]

so, when you multiply two complex numbers, the phases add:

\[ e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}. \]
## Properties of complex numbers

### Given
\[ z = a + ib \]
\[ w = c + id \]

### Operations

**Add**
\[ z + w = (a + c) + i(b + d) \]

**Multiply**
\[ zw = (a + ib)(c + id) \]
\[ = ab + iad + ibc + i^2bd \]
\[ = (ab - bd) + i(ad + bc) \]

**Real part**
\[ \text{Re}(z) = a \]

**Imaginary part**
\[ \text{Im}(z) = b \]

**Absolute value**
\[ |z| = \sqrt{a^2 + b^2} \]

**Conjugate**
\[ z^* = a - ib \]

*(Exercise: Show that \(zz^* = |z|^2\))*